TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 196, 1974

MAYER-VIETORIS SEQUENCES AND BRAUER GROUPS OF NONNORMAL DOMAINS

BY

L. N. CHILDS[1]

ABSTRACT. Let R be a Noetherian domain with finite integral closure $\overline{R}$. We study the map from the Brauer group of R, $B(R)$, to $B(\overline{R})$: first, by embedding $B(R)$ into the Čech etale cohomology group $H^2(R, U)$ and using a Mayer-Vietoris sequence for Čech cohomology of commutative rings; second, via Milnor's theorem from algebraic K-theory. We apply our results to show, i.e., that if R is a domain with quotient field K a global field, then the map from $B(R)$ to $B(K)$ is 1-1.

Let R be a Noetherian integral domain with finite integral closure $\overline{R}$, conductor c and quotient field K. The object of this paper is to try to describe relationships between the Brauer group of R, $B(R)$, and $B(\overline{R})$, $B(R/c)$, and $B(K)$. Questions of this kind were considered by M. Auslander and O. Goldman, who showed that if $R = \overline{R}$ is regular, the map from $B(R)$ to $B(K)$ is 1-1.

Our approach in the first three sections is to glean information from a long exact Mayer-Vietoris sequence of Čech cohomology. This sequence extends a six-term Mayer-Vietoris K-theory sequence for the category Pic of Milnor and Bass, and when $B(R)$ is isomorphic to the second etale cohomology group with coefficients in the sheaf of units (multiplicative group) the extended sequence describes the kernel and image of the map from $B(R)$ to $B(\overline{R}) \oplus B(R/c)$. In particular, for R of dimension 1 the kernel is trivial.

When R has dimension 1, $\overline{R}$ is regular, so the only candidates for elements in the kernel of the map from $B(R)$ to $B(K)$ are elements of $B(R)$ which become trivial in $B(\overline{R})$ but not in $B(R/c)$. Auslander and Goldman's counterexample to $B(R) \longrightarrow B(K)$ being 1-1 is of this kind. As a consequence it follows that if R is any ring with quotient field K a global field, the map $B(R) \longrightarrow B(K)$ is 1-1. We get the following splitting result: If A is any Azumaya R-algebra, R a ring with quotient field a global field K, and $A \otimes_R K$ is split by a finite extension field L, then every order over R in L splits A.

When $B(R)$ cannot be identified cohomologically, cohomological methods do not give precise information on the kernel of the map $B(R) \longrightarrow B(\overline{R}) \oplus B(R/c)$. In

Received by the editors January 31, 1972 and, in revised form, October 15, 1973.
AMS (MOS) subject classifications (1970). Primary 13A20; Secondary 13B20, 14F20.
(1) Partially supported by the National Science Foundation Grants GP 29652 XI and SD GU 3171.

§4 we apply methods of Milnor and Bass to obtain six-term Mayer-Vietoris K-theory sequences for the categories **FP** and **Az**. Using them we show that when $\mathrm{Pic}(\overline{R}/c)$ is torsion, the kernel is isomorphic to the cokernel of the map from $\mathrm{Pic}(\overline{R}) \oplus \mathrm{Pic}(R/c)$ to $\mathrm{Pic}(\overline{R}/c)$.

I wish to thank E. D. Davis for assistance with commutative algebra in §3 and M. Ojanguren for pointing out an error in §2.

I. Let R be a commutative ring, and let $\mathbf{K} = \mathbf{K}(R)$ be the category of commutative R-algebras and R-algebra maps. Given a functor $F\colon \mathbf{K} \to \mathbf{ab}$ and an R-algebra S define $FS\colon \mathbf{K} \to \mathbf{ab}$ by $FS(T) = F(S \otimes T)$ (unadorned $\otimes$ means $\otimes_R$); define $FS/F\colon \mathbf{K} \to \mathbf{ab}$ by exactness of the sequence

$$F(T) \to FS(T) \to FS/F(T) \to 0$$

for all objects T of $\mathbf{K}$. (The functor FS/F is called $QF(S, -)$ in [6].)

Denote by $H^n(T/R, F)$ the Amitsur cohomology groups with coefficients in F. Then $H^n(T/R, FS) \cong H^n(T \otimes S/S, F)$, the isomorphism being induced on the complex level. For an R-algebra T, $T^n = T \otimes \cdots \otimes T$ (n times), $T^0 = R$.

(1.1) **Proposition** [6, 3.3]. *If S, T are R-algebras and $F\colon \mathbf{K} \to \mathbf{ab}$ a functor such that $F(T^n) \to FS(T^n)$ is 1-1 for all $n \geq 0$, then there is a long exact sequence*

$$\cdots \to H^{n-1}(T/R, FS/F) \to H^n(T/R, F) \to H^n(T/R, FS)$$
$$\to H^n(T/R, FS/F) \to H^{n+1}(T/R, F) \to \cdots.$$

The proof is a routine diagram chase.

We remark that with $U\colon \mathbf{K} \to \mathbf{ab}$ the units functor the group $H^1(T/R, US/U)$ has been studied in [6, §4] and [8].

(1.2) Let $\mathbf{T}$ be a (Grothendieck) topology on R for which all the covers are singleton covers which are faithfully flat R-algebras (I am viewing $\mathbf{T}$ as involving algebras, rather than affine schemes, so a singleton cover is an R-algebra map $S \to T$). Assume also that if $R \to S$ is a cover in $\mathbf{T}$, so is $R \to S^n$ for all $n > 0$. Denote

$$\varinjlim H^n(T/R, F) = H^n_{\mathbf{T}}(R, F),$$
$$\varinjlim H^n(T/R, FS) = H^n_{\mathbf{T}}(S, F),$$
$$\varinjlim H^n(T/R, FS/F) = H^n_{\mathbf{T}}(S, R; F),$$

where the limits are taken over covers $R \to T$ in $\mathbf{T}$. Thus $H^n_{\mathbf{T}}(S, F)$, where $\mathbf{T}$ is a topology on R, is a limit over covers of S of the form $S \to S \otimes_R T$. (In case $\mathbf{T}$ consists of covers which are faithfully flat of finite presentation these limits are denoted by H^nF, $H^nF(S)$, $H^nQF(S)$ in [6].)

(1.3) **Proposition.** *If $F\colon \mathbf{K} \to \mathbf{ab}$ is a functor for which for all covers $R \to T$ in $\mathbf{T}$, $F(T) \to FS(T)$ is 1-1, then there is a long exact sequence*

$$\cdots \to H^{n-1}_{\mathbf{T}}(S, R; F) \to H^n_{\mathbf{T}}(R, F) \to H^n_{\mathbf{T}}(S, F)$$
$$\to H^n_{\mathbf{T}}(S, R; F) \to H^{n+1}_{\mathbf{T}}(R, F) \to \cdots .$$

The groups $H^n_{\mathbf{T}}(S, R; F)$ are candidates for an excision property. Let

(1.4)
$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \xrightarrow{f_2} & R_3 \end{array}$$

be a fibre product diagram in $\mathbf{K}$, so that $R = \{(r_1, r_2) | f_1(r_1) = f_2(r_2)\}$.

(1.5) **Definition** (cf. [10]). Given (1.4) with f_2 onto, and a topology $\mathbf{T}$ over R as in (1.3), a functor $F: \mathbf{K} \to \mathbf{ab}$ is a M-V functor (over $\mathbf{T}$ relative to (1.4)) if

(MV 1) $$0 \to F(S) \to F(S_1) \oplus F(S_2) \xrightarrow{\rho_S} F(S_3)$$

is exact for all $R \to S$ in $\mathbf{T}$, where $\rho_S = F(S \oplus f_1) - F(S \oplus f_2)$, and

(MV 2) $$\varinjlim \rho_T \text{ is onto;}$$

i.e. given $R \to S$ in $\mathbf{T}$ and $x \in F(S_3)$, there exists $S \xrightarrow{i} T$ in T so that $F(i \otimes R_3)(x)$ is in the image of ρ_T.

We can now state an excision theorem.

(1.6) **Theorem.** *Let*

(1.4)
$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \xrightarrow{f_2} & R_3 \end{array}$$

be a fibre product diagram with f_2 onto, and $F: \mathbf{K}(R) \to \mathbf{ab}$ a functor. Suppose $\mathbf{T}$ is a topology over R as in (1.2), *and F is a M-V functor over $\mathbf{T}$ relative to* (1.4). *Then for all $n \geq 0$, $H_{\mathbf{T}}(R_2, R, F) \cong H_{\mathbf{T}}(R_3, R_1, F)$.*

Proof. For $n \geq 0$ and T a cover of R in $\mathbf{T}$ consider the commutative diagram

$$\begin{array}{ccccccc} F(T^n) & \to & F(T_2^n) & \xrightarrow{\rho'_T} & F(T_2^n)/F(T^n) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow \pi_T^n & & \\ F(T_1^n) & \to & F(T_3^n) & \longrightarrow & F(T_3^n)/F(T_1^n) & \to & 0 \end{array}$$

which has exact rows. Since F satisfies (MV 1) a quick diagram chase shows that π^n_T must be 1-1. Assume F satisfies (MV 2). Given an element z of $F(T_3^n)/F(T_1^n)$ pull it back to w in $F(T_3^n)$. By (MV 2) there is a cover $T \xrightarrow{i} U$ in $\mathbf{T}$ so that $F(i \otimes R_3)(w) = w_U = F(f_2 \otimes U^n)(y) - F(f_1 \otimes U^n)(x)$. But then $\rho_U(w_U) = \rho_U(F(f_2 \otimes U^n)(y))$. So $\rho'_U(y)$ in $F(U_2^n)/F(U^n)$ maps onto z_U. It is then a trivial

3-dimensional diagram chase to verify that

$$\varinjlim H(\pi^n_T): H^n(R_2, R, F) \to H^n(R_3, R_1, F)$$

is 1-1 and onto.

(1.7) **Corollary.** *Given the fibre product* (1.4) *with* f_2 *onto*, $\mathbf{T}$ *a topology and* F *a functor*: $\mathbf{K} \to \mathbf{ab}$ *satisfying*

(1) $F(T_i) \to F(T_{i+2})$ *is* 1-1 *for all* $R \to T$ *in* $\mathbf{T}$,

(2) F *is a M-V functor over* $\mathbf{T}$ *relative to* (1.4).

Then there is a Mayer-Vietoris sequence

$$\cdots \to H^{n-1}_{\mathbf{T}}(R_3, F) \to H^n_{\mathbf{T}}(R, F)$$
$$\to H^n_{\mathbf{T}}(R_1, F) \oplus H^n_{\mathbf{T}}(R_2, F) \to H^n_{\mathbf{T}}(R_3, F) \to \cdots.$$

This follows by a well-known and easy diagram chase on the long exact sequences of (1.3) involving $H^n_{\mathbf{T}}(R_i, F) \to H^n_{\mathbf{T}}(R_{i+2}, F)$, $i = 0, 1$ ($R_0 = R$), using the excision isomorphism of (1.6).

(1.8) **Remark.** One can prove (1.7) without the assumption (1) by a more tedious direct argument, not using the excision property (1.6). We omit a proof of this, as the added generality is not needed in the applications below. A reader wishing to study the Brauer group of the group ring over Z of a cyclic group of prime power order using [3, p. 483, (5.5) and p. 601], however, will prefer the more general version of (1.7).

Recall that U: $\mathbf{K} \to \mathbf{ab}$ is the units functor, Pic: $\mathbf{K} \to \mathbf{ab}$ is the functor which assigns to $\mathbf{T}$ the group of isomorphism classes of rank one projective T-modules, and B: $\mathbf{K} \to \mathbf{ab}$ is the Brauer group functor.

For x in R, R_x denotes the ring of quotients of R with respect to the multiplicative set $\{x^n | n \geq 0\}$. An R-algebra T of the form $T = \bigoplus_{i=1}^n R_{x_i}$ with $Rx_1 + \cdots + Rx_n = R$ will be called a Zariski cover of R. Such an R-algebra is faithfully flat over R [5, p. 137].

(1.9) **Proposition.** *Let* $\mathbf{T}$ *be a topology over* R *as in* (1.2) *such that* (*) *for* $R \to S$ *a cover in* $\mathbf{T}$ *and* $x \in \mathrm{Pic}(S)$ *there exists* $S \to T$ *a cover of* $\mathbf{T}$ *such that* $x \in \ker\{\mathrm{Pic}(S) \to \mathrm{Pic}(T)\}$. *Then the functor* U *is a M-V functor over* $\mathbf{T}$ *relative to* (1.4) (*when* f *is onto*).

Proof. That U satisfies (MV 1) is clear. That U satisfies (MV 2) follows from a Mayer-Vietoris sequence of Milnor [3, p. 481]: surjectivity of f_2 in (1.4) yields the exact sequence

$$0 \to U(S) \to U(S_1) \oplus U(S_2) \to U(S_3) \xrightarrow{\partial} \mathrm{Pic}(S)$$

for any flat R-algebra S. Given any $x \in U(S_3)$ find a cover i: $S \to T$ which splits $\partial(x)$ in $\mathrm{Pic}(S)$. Then $U(i \otimes R_3)(x)$ comes from $U(T_1) \oplus U(T_2)$, so that (MV 2) holds.

Remark. We show in §2 that the topology whose covers $S \to T$ are etale faithfully flat algebras satisfies (*).

(1.10) **Corollary.** *Let* $\mathbf{T}$ *be a topology over* R *as in* (1.2) *and satisfying* (*). *Suppose given the fibre product diagram*

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \xrightarrow{f_2} & R_3 \end{array} \tag{1.4}$$

with f_2 *onto, and suppose that*

$$\text{for each maximal ideal } m \text{ of } R, \quad R_m \otimes R_i \text{ is semilocal, } i = 1, 2, 3. \tag{1.11}$$

Then there is a long exact sequence

$$0 \to U(R) \to U(R_1) \oplus U(R_2) \to U(R_3) \to \mathrm{Pic}(R) \to \mathrm{Pic}(R_1) \oplus \mathrm{Pic}(R_2)$$
$$\to \mathrm{Pic}(R_3) \to H^2_{\mathbf{T}}(R, U) \to H^2_{\mathbf{T}}(R_1, U) \oplus H^2_{\mathbf{T}}(R_2, U) \to H^2_{\mathbf{T}}(R_3, U)$$
$$\to H^3_{\mathbf{T}}(R, U) \to \cdots .$$

Proof. Apply (1.9) to the sequence (1.7) (noting remark (1.8)). The result then follows once we note that $H^0_{\mathbf{T}}(R_i, U) = U(R_i)$ and $H^1_{\mathbf{T}}(R_i, U) \cong \mathrm{Pic}(R_i)$. The first of these is true because for any flat R-algebra T, $H^0(T/R, U) = U(R)$. As for the second, given any rank one projective R_i-module P_i and any maximal ideal m of R, $R_m \otimes_R P_i$ is free because $R_m \oplus_R R_i$ is semilocal. Thus for each m there is an x not in m so that $R_x \otimes_R R_i$ is free. The x's generate the unit ideal so there exists $x_1, \cdots, x_n$ so that $x_1 + \cdots + x_n = 1$, and $(\bigoplus_{j=1}^n R_{x_j}) \otimes_R P_i$ is free. But such an algebra $T = \bigoplus R_{x_j}$ is a Zariski cover. Thus the class of P_i is in

$$\ker(\mathrm{Pic}(R_i) \to \mathrm{Pic}(T \otimes R_i)) \cong H^1(T \otimes R_i/R_i, U).$$

Passing to the limit (union) over covers of R_i in $\mathbf{T}$ yields the isomorphism $H^1_{\mathbf{T}}(R_i, U) \cong \mathrm{Pic}(R_i)$ [6, 6.6].

(1.12) **Remark.** The first six terms of this sequence form the sequence of Milnor used in the proof of (1.9). (Milnor's sequence is however valid without assumption (1.11).) Corollary (1.10) thus yields, under the condition (1.11), an infinite extension of Milnor's sequence.

If $\mathbf{T}$ is the etale topology (see below) $H^2_{\mathbf{T}}(R, U)$ is isomorphic to Grothendieck's cohomological Brauer group [11] by a result of M. Artin [17, Corollary 4.2], and contains the usual Brauer group. Corollary (1.10) thus gives information on the cohomological Brauer group of a fibre product.

II. In this section we study the relationship between the usual Brauer group of R and the Čech Brauer group which arose in §I.

In the following sections R is a commutative ring and $\mathbf{T}$ is the topology on R such that $\mathrm{Cat}(\mathbf{T})$ = R-algebras $R \to S$ when $R \to S$ is in $\mathrm{Cov}(\mathbf{T})$, and $\mathrm{Cov}(\mathbf{T})$ = algebras $S \to T$ where T is a finitely presented, faithfully flat etale S-algebra [14]. If R' is an R-algebra, the topology $\mathbf{T}(R)$ on R' is the topology $\mathbf{T} \otimes_R R'$, i.e.

$$\mathrm{Cat}(\mathbf{T} \otimes_R R') = \{R'\text{-algebras } S' = S \otimes_R R'\},$$

$$\mathrm{Cov}(\mathbf{T} \otimes R') = \{\text{covers } S \otimes R' \to T \otimes R' \text{ with } S \to T \text{ in } \mathrm{Cov}(\mathbf{T})\}.$$

Call $\mathbf{T}$ the etale topology on R, and $S \to T \in \mathrm{Cov}(\mathbf{T})$ an etale cover.

The object of this section is to prove:

(2.1) **Theorem.** *Let R_0 be a commutative Noetherian ring, R a finite R_0-algebra, $\mathbf{T} = \mathbf{T}(R_0) \otimes R$. Let $B_{\mathbf{T}}(R) = \bigcup \ker(B(R) \to B(T))$ where the union runs through covers $R \to T$ in $\mathbf{T}$. Then there is a 1-1 map $B_{\mathbf{T}}(R) \to H^2_{\mathbf{T}}(R, U)$.*

The idea of the proof is the same as that for the corresponding map into $H^2_{\mathbf{T}}(R, U)$, the derived sheaf cohomology, as found in [11, I, §§1.3 and 2]. Before beginning the proof, we need:

(2.2) **Proposition.** *Let $\bar{R}$ be a finite R-algebra, let S be an etale cover of R, let $\bar{S} = S \otimes_R \bar{R}$, and let P be a finite $S \otimes_{\bar{R}} \bar{S}$-module of rank n. Then there exists an etale cover $S \to T$ such that if $\bar{T} = T \otimes_R \bar{R}$, then $P \otimes_{(\bar{S} \otimes_{\bar{R}} \bar{S})} (\bar{T} \otimes_{\bar{R}} \bar{T})$ is free.*

Proof. Let q be a prime ideal of $S \otimes_R S$. Then $R \otimes_R (S \otimes_R S)_q$ is semi-local, so $\bar{P} \otimes_{\bar{S}^2} (\bar{R} \otimes_R (S \otimes_R S)_q)$ is free. Thus there is an etale (in fact a Zariski) cover $W = \bigoplus_{r=1}^n (S \otimes_R S)_{f_i}$ of $S \otimes_R S$ so that $P \otimes_{\bar{S}^2} (\bar{R} \otimes_R W)$ is free. By [17, Theorem 4.1], since R is Noetherian it follows that there is an etale S-algebra T such that the map $S \otimes_R S \to T \otimes_R T$ factors through the map $S \otimes_R S \to W$. Thus

$$P \otimes_{\bar{S}^2} (\bar{R} \otimes_R (T \otimes_R T)) \cong P \otimes_{\bar{S}^2} (\bar{T} \otimes_{\bar{R}} \bar{T})$$

is free.

Proof (of (2.1)). Let $GL_\otimes = \varinjlim GL_\otimes(n)$, where the limit is taken with respect to the maps $GL(n) = \mathrm{Aut}(R^n) \to \mathrm{Aut}(R^n \otimes R^m) = GL(mn)$ by $\sigma \to \sigma \otimes 1$, and let $PGL = \varinjlim PGL(n)$, the limit being taken in the same way. Then there is a short exact sequence of functors $1 \to U \to GL_\otimes \to PGL \to 1$ which yields an exact sequence of pointed Čech cohomology sets:

$$(2.2) \qquad \cdots H^1_{\mathbf{T}}(R, U) \to H^1_{\mathbf{T}}(R, GL_\otimes) \to H^1_{\mathbf{T}}(R, PGL) \to H^2_{\mathbf{T}}(R, U).$$

But in fact we have an abelian monoid structure on $H^1_{\mathrm{T}}(R, GL_{\otimes})$ and on $H^1_{\mathrm{T}}(R, PGL)$ given as follows: Recall that

$$H^1_{\mathrm{T}}(R, GL_{\otimes}) = \varinjlim H^1(S/R, GL_{\otimes}) \quad \text{and} \quad H^1(S/R, GL_{\otimes}) = \varinjlim_n H^1(S/R, GL(n)).$$

For $\bar{a}$, $\bar{b}$ in $H^1(S/R, GL_{\otimes})$ let

$$a \in Z^1(S/R, GL(n)) \subseteq GL_n(S^2), \qquad b \in Z^1(S/R, GL(m)) \subseteq GL_m(S^2)$$

be preimages. Then $\bar{a} \times \bar{b}$ is the image in $H^1(S/R, GL_{\otimes})$ of $a \otimes 1 \cdot 1 \otimes b$ in

$$Z(S/R, GL(mn)) \subseteq GL_{mn}(S^2) = \mathrm{Aut}((S \otimes S)^{(n)} \otimes (S \otimes S)^{(m)}).$$

It is shown in [9, p. 107] that this multiplication and the analogous multiplication on $H^1(S/R, PGL)$, is well defined, commutative, and makes these sets into abelian monoids. Garfinkel [9] also shows that the maps of (2.2) are homomorphisms of abelian monoids.

In fact, $H^1_{\mathrm{T}}(R, GL_{\otimes})$ and $H^1_{\mathrm{T}}(R, PGL)$ are abelian groups, as we shall see. Now descent theory ([12], [9]) yields a natural (in S and n) bijection of pointed sets $H^1(S/R, GL(n)) \leftrightarrow V(S/R, S^n)$ where $V(S/R, S^n)$ is the set of isomorphism classes of R-modules P which are projective of rank n and such that $P \otimes_R S \cong S^n$. Passing to the limit over n gives a bijection of $H^1(S/R, GL_{\otimes})$ with the set of those isomorphism classes of projective R-modules P of finite constant rank over R, which became free when tensored with S, modulo the equivalence relation that $P \sim Q$ if $P \otimes F_1 \cong Q \otimes F_2$ for some free modules F_1, F_2 of finite rank. The abelian monoid structure on $H^1(S/R, GL_{\otimes})$, then coincides with the tensor product on projective modules.

Passing to the limit over $R \to S$ in $\mathbf{T}$ gives an abelian monoid isomorphism of $H^1_{\mathrm{T}}(R, GL_{\otimes})$ with the monoid of equivalence classes of projective R-modules of finite constant rank. But by a theorem of Bass, for any projective module P of finite rank there is another such Q so that $P \otimes Q$ is free. Thus $H^1_{\mathrm{T}}(R, GL_{\otimes})$ is an abelian group. If $\mathbf{FP}_r(R)$ denotes the category (with product $\otimes$) of the projective R-module of constant finite rank, then in fact

$$H^1_{\mathrm{T}}(R, GL_{\otimes}) \cong K_0\mathbf{FP}_r/\mathbf{Z} = (\mathrm{def})\, \tilde{K}_0\mathbf{FP}_r(R)$$

where $\mathbf{Z}$ denotes the classes of the free R-modules of finite rank.

Let $AE(n)$ be the functor defined by $AE(n)(S) = \mathrm{Aut}(\mathrm{End}_S(S^{(n)}))$ and $AE = \varinjlim AE(n)$; the limit being induced by the maps $S^{(n)} \rightarrowtail S^{(n)} \otimes S^{(m)}$ just as with $GL_{\otimes}$ and PGL. Then $H^1(S/R, AE)$ is an abelian monoid [9, 6.13, p. 109]. There is an exact sequence of functors [15]

$$1 \to PGL(n) \to AE(n) \to \ker(\mathrm{Pic} \xrightarrow{n} \mathrm{Pic}) \to 1$$

which, in the limit, yields $1 \to PGL \to AE \to t\,\mathrm{Pic} \to 1$ where $t\,\mathrm{Pic}$ is the torsion subgroup of Pic. Applying cohomology yields the exact sequence

$$\to H^0_{\mathbf{T}}(R, t\,\mathrm{Pic}) \to H^1_{\mathbf{T}}(R, PGL) \to H^1_{\mathbf{T}}(R, AE) \to H^1_{\mathbf{T}}(R, t\,\mathrm{Pic})$$

which is an exact sequence of abelian monoids. But in fact $H^0_{\mathbf{T}}(R, t\,\mathrm{Pic}) \subseteq \varinjlim_{\mathbf{T}} t\,\mathrm{Pic}(S) = 0$ and $H^1_{\mathbf{T}}(R, \mathrm{Pic}_t) \subseteq \varinjlim_{\mathbf{T}} t\,\mathrm{Pic}(S^2) = 0$ since for P in $t\,\mathrm{Pic}(S^2)$ there is a cover $S \xrightarrow{i} T$ so that $P \otimes_{S^2} T^2 \cong T^2$ by Proposition (2.2). Thus $H_{\mathbf{T}}(R, PGL) \cong H_{\mathbf{T}}(R, AE)$.

Now descent theory yields a natural (in S and n) bijection of pointed sets. $H^1(S/R, AE(n)) \to V(S/R, \mathrm{End}_s(S^n))$ where $V(S/R, \mathrm{End}_s(S^n))$ is the set of isomorphism classes of Azumaya R-algebras A such that $A \otimes_R S \cong \mathrm{End}_s(S^n)$. Passing to the limit over n gives a bijection of $H^1(S/R, AE)$ with the set of Azumaya R-algebras A of constant rank over R which become isomorphic to matrix rings when tensored with S, modulo the equivalence relation that $A \approx B$ if $A \otimes M_n(R) \cong B \otimes M_m(R)$ for some matrix rings $M_n(R)$, $M_m(R)$, $m, n \geq 0$. The abelian monoid structure on $H^1(S/R, AE)$ coincides with the tensor product of Azumaya algebras.

Passing to the limit over $R \to S$ in $\mathbf{T}$ gives an abelian monoid isomorphism of $H^1_{\mathbf{T}}(R, AE)$ with the monoid of equivalence classes of Azumaya R-algebras of constant rank which are split into matrix rings by covers of $\mathbf{T}$. But this monoid is a group, for if A is split by a cover in $\mathbf{T}$, so is A^0, and if Q is a projective R-module of constant rank such that $Q \otimes_R A$ is a free R-module, then Q is split by some cover in $\mathbf{T}$, so that $A^0 \otimes \mathrm{End}_R(Q)$ is split by a cover of $\mathbf{T}$. But $A \otimes A^0 \otimes \mathrm{End}_R(Q) =$ a matrix ring, so A has an inverse which is split in $\mathbf{T}$.

If $\mathrm{Az}^r_{\mathbf{T}}(R)$ denotes the category with product $\otimes$ of the Azumaya R-algebras of finite constant rank split by $\mathbf{T}$, then as above we have

$$H^1_{\mathbf{T}}(R, AE) \cong K_0(\mathrm{Az}^r_{\mathbf{T}}(R)/\mathbf{Z}) = (\mathrm{def})\, \tilde{K}_0\mathrm{Az}^r_{\mathbf{T}}$$

where $\mathbf{Z}$ denotes the classes of the endomorphism rings of free R-modules of finite rank.

Putting these identifications together, we have the commutative diagram with vertical isomorphisms and exact rows:

(2.3)
$$\begin{array}{ccccccccc}
\to H^1_{\mathbf{T}}(R, U) & \to & H^1_{\mathbf{T}}(R, GL_{\otimes}) & \to & H^1_{\mathbf{T}}(R, PGL) & \to & H^2_{\mathbf{T}}(R, U) & & \\
\updownarrow & & \updownarrow & & \updownarrow & & \uparrow & & \\
\mathrm{Pic}(R) & \to & \tilde{K}_0\mathrm{FP}_r(R) & \xrightarrow{\mathrm{End}} & \tilde{K}_0\mathrm{Az}^r_{\mathbf{T}}(R) & \longrightarrow & B & \to & 0
\end{array}$$

It follows that if B is defined by exactness of the bottom row there is a monomorphism from B into $H^2_{\mathbf{T}}(R, U)$. But it is easy to see that $B = B_{\mathbf{T}}(R)$. For B consists of Azumaya algebras of constant rank modulo the equivalence relation $A \sim A'$ if there exist projective modules with constant rank, P, P' so that $A \otimes \mathrm{End}_R(P) \cong A \otimes \mathrm{End}_R(P')$. But any class in $B_{\mathbf{T}}(R)$ may be represented by an Azumaya algebra of constant rank, so the obvious 1-1 map $B \to B_T(R)$ is onto.

This completes the proof of Theorem (2.1).

III.

(3.1) Let R be a Noetherian domain, $\overline{R}$ its normalization, c the conductor. Then there is a long exact sequence.

$$1 \to U(R) \to U(\overline{R}) \to U(\overline{R}/c) \to U(R/c) \to \mathrm{Pic}\,(R) \to \mathrm{Pic}\,(\overline{R}) \oplus \mathrm{Pic}\,(R/c)$$
$$\to \mathrm{Pic}(\overline{R}/c) \to H^2_T(R, U) \to H^2_T(\overline{R}, U) \oplus H^2_T(R/c, U) \to H^2_T(\overline{R}/c, U)$$
$$\to H^3_T(R, U) \to \cdots .$$

Proof. This follows immediately from Corollary (1.10) applied to the fibre product

$$\begin{array}{ccc} R & \to & R/c \\ \downarrow & & \downarrow \\ \overline{R} & \to & \overline{R}/c \end{array}$$

since for each prime ideal p of R, $\overline{R}_p$ is semilocal [13, (33.10)].

Note that this sequence is nontrivial iff $c \neq 0$ iff $\overline{R}$ is a finite R-module.

The rest of the section is devoted to applications of (3.1).

(3.2) **Theorem.** *Suppose R is a Noetherian domain of Krull dimension 1, then there is an exact sequence*

$$0 \to B(R) \to B_T(\overline{R}) \oplus B_T(R/c) \to B_T(\overline{R}/c).$$

Proof. The conditions on R imply that $B(R) \cong dH^2_T(R, U)$, the derived sheaf cohomology by [11, II, Corollary 2.2]. Artin's isomorphism of derived and Čech etale cohomology [17, Corollary 4.2] implies that $B(R) \cong H^2_T(R, U)$. The other three Brauer groups map monomorphically into the respective H^2's by (2.1). Since $\overline{R}$ is Dedekind, $\overline{R}/c$ is Artinian, so $H^1_T(\overline{R}/c, U) \subseteq \mathrm{Pic}\,(\overline{R}/c) = 0$. So exactness of the sequence follows easily from (3.1).

(3.3) **Corollary.** *Let R be any ring with quotient field a finite algebraic number field K. Then* $B(R) \cong B(\overline{R})$.

Proof. By [7, Theorem A] $\overline{R}$ is a ring of quotients of the integral closure R_0 of $\mathbf{Z}$ in $\overline{R}$. Thus R_0, hence $\overline{R}$, hence R, has dimension 1. Since R_0 is a finite $\mathbf{Z}$-module, R is pseudo-geometric [13] so $\overline{R}$ is a finite R-module. Thus $c \neq 0$ and $\overline{R}/c$, hence also R/c are finite rings. Thus $B(R/c) = B(\overline{R}/c) = 0$ [12, p. 41, Corollary 3]. The result then follows from (3.2).

(3.4) **Corollary.** *If R is any ring with quotient field a finite algebraic number field K, then the map $B(R) \to B(K)$ is 1-1.*

Proof. Since $\overline{R}$ is normal of dimension 1, $\overline{R}$ is regular so $B(\overline{R}) \longrightarrow B(K)$ is 1-1 by [2, 7.2].

(3.5) **Proposition.** *Let k be a field, R a k-algebra which is a Noetherian domain with quotient field K a finitely generated function field of transcendence degree ≤ 1 over k. Then there is an exact sequence*

$$0 \longrightarrow B(R) \longrightarrow B_T(\overline{R}) \oplus B_T(R/c) \longrightarrow B_T(\overline{R}/c).$$

Proof. If tr deg $K = 0$ then $R = K$ and the sequence is trivial. If tr deg $K = 1$ then R contains a transcendental element x, hence contains $k[x]$, and K is a finite extension of $k(x)$. In that case it is known that R has dimension 1 (e.g. [7, §3]).

(3.6) **Corollary.** *If in (3.5) k has cohomological dimension ≤ 1* [16] *then $B(R) \cong B_T(\overline{R})$, and the map from $B(R)$ to $B(K)$ is* 1-1.

Proof. Assume tr deg $K = 1$. We show that $\overline{R}$ is a finite R-module. Let x be an element of R which is transcendental over k, let Z be the integral closure of $k[x]$ in R, and let $\overline{Z}$ be the integral closure of Z in K. Then $\overline{Z} = Z[a_1, \cdots, a_n]$, for $k[x]$ is pseudo-geometric [13] and K is a finite extension of $k(x)$, so $\overline{Z}$ is a finite $k[x]$-module. *Claim*: $\overline{R} = R[a_1, \cdots, a_n]$. Clearly since $\overline{Z} \subseteq \overline{R}$, $R[a_1, \cdots, a_n] \subseteq \overline{R}$. On the other hand, for any maximal ideal p of $\overline{Z}$, $\overline{Z}_p \subseteq R[a_1, \cdots, a_n]_p \subseteq \overline{R}_p$, so since $\overline{Z}_p$ is a discrete rank one valuation ring, $\overline{Z}_p = \overline{R}_p$. Thus if $\tilde{Z} = \bigoplus_p \overline{Z}_p$, $R[a_1, \cdots, a_n] \otimes_Z \tilde{Z} = \overline{R} \otimes_Z \tilde{Z}$. Since $\tilde{Z}$ is a faithfully flat Z-module, $R[a_1, \cdots, a_n] = \overline{R}$.

It follows that $c \neq 0$.

Since $\overline{R}$ is Dedekind, $\overline{R}/c$ and $R/c \subseteq \overline{R}/c$ are finitely generated k-algebras of dimension 0. Thus $\overline{R}/c$ is a finite k-algebra, so that $(\overline{R}/c)/\text{rad}(\overline{R}/c)$ is a finite product of finite field extensions of k. Since every finite extension of k has trivial Brauer group, $B((\overline{R}/c)/\text{rad}(\overline{R}/c)) = 0$. Now since $\text{rad}(\overline{R}/c)$ is nilpotent, it follows from [12, p. 41] that the map $B(\overline{R}/c) \longrightarrow B((\overline{R}/c)/\text{rad}(R/c))$ is 1-1. Thus $B(\overline{R}/c) = 0$. Similarly $B(R/c) = 0$. The result follows from (3.2) and [2, 7.2], as before.

This last corollary applies in particular if k is algebraically closed, or if k is a finite field (see [16] for other examples). Thus

(3.7) **Corollary.** *If R is any ring with quotient field a global field K, the map $B(R) \longrightarrow B(K)$ is* 1-1.

(3.8) **Corollary** [11, III, (1.2)]. *If R is the affine ring of any complex algebraic curve, then* $B(R) = 0$.

(Tsen's theorem implies that $B(K) = 0$.)

We make a remark about splitting algebras.

(3.9) **Corollary.** *Let R be a ring with quotient field K a global field. Let A*

be an Azumaya R-algebra and $L \supseteq K$ *be a finite splitting field for* $A \otimes_R K$. *Then every order over* R *in* L *splits* A.

For if S is such an order, $B(S) \to B(L)$ is 1-1.

We conclude this section by observing that Auslander and Goldman's well-known counterexample to $B(R) \to B(K)$ being 1-1 may be described in terms of (3.2).

(3.10) **Example** (**Auslander-Goldman**) [2, p. 388], [11, II, p. 297–09]. Let $R = \mathbb{R}[x, y]$ with $x^2 + y^2 = 0$. Then $\overline{R} = \mathbb{R}[x, y/x] \cong \mathbb{C}[x]$, $c = xR$, $R/c = \mathbb{R}$, $\overline{R}/c \cong \mathbb{C}$. We have $B(\overline{R}/c) = 0$; $B(\overline{R}) = 0$ by [2, 7.2] and Tsen's theorem; and $B(R/c) = B(\mathbb{R}) \cong \mathbf{Z}/2\mathbf{Z}$. So by (3.2) $B(R) \cong B_{\mathrm{T}}(R/c) \subseteq B(R/c)$. But in fact $B_{\mathrm{T}}(R/c) = B(R/c)$, for the cover $R \to R \otimes_{\mathrm{R}} \mathbb{C}$ in T splits the nontrivial element of $B(R/c)$. So $B(R) \cong B(R/c) = \mathbf{Z}/2\mathbf{Z}$.

Thus $B(R) \to B(\overline{R})$ is not 1-1, and so the map from the Brauer group of R to the Brauer group of its quotient field is not 1-1.

IV. Sequence (3.1) describes the kernel of the map from the (Čech) cohomological Brauer group $B'(R)$ to $B'(\overline{R}) \otimes B'(R/c)$ in terms of $\mathrm{Pic}(\overline{R}/c)$. When $B'(R) \cong B(R)$ as when R has dimension 1, this gives information on the kernel of the map from $B(R)$ to $B(\overline{R}) \oplus B(R/c)$ but in general (3.1) only describes that kernel as a quotient of some subgroup of $\mathrm{Pic}(\overline{R}/c)$.

The object of this section is to prove that $t\,\mathrm{Pic}(\overline{R}/c)$ always maps into the kernel. We prove this by obtaining Mayer-Vietoris K-theory sequences for the categories **Fp** and **Az** (using results of Bass and Milnor) and chasing the diagram which arises from the map from one of the sequences to the other.

If $\mathrm{Pic}(\overline{R}/c)$ is torsion we can then describe the kernel precisely:

(4.1) **Theorem.** *Let*

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \xrightarrow{f_2} & R_3 \end{array}$$

be a fibre product of rings with f_2 *onto. Then if the map* $\mathrm{Pic}(R_1) \oplus \mathrm{Pic}(R_2) \to \mathrm{Pic}(R_3)/t\,\mathrm{Pic}(R_3)$ *is onto, then there is an exact sequence*

(4.1a) $$t\,\mathrm{Pic}(R_3) \to B(R) \to B(R_1) \oplus B(R_2).$$

If $\mathrm{Pic}(R_3)$ *is torsion, then there is an exact sequence* $\cdots \to \mathrm{Pic}(R) \to \mathrm{Pic}(R_1) \oplus \mathrm{Pic}(R_2) \to \mathrm{Pic}(R_3) \to B(R) \to B(R_1) \oplus B(R_2)$.

Recall the categories with product $\mathrm{FP}(R)$ and $\mathrm{Az}(R)$ for R a commutative ring. The objects of $\mathrm{FP}(R)$ are finitely generated projective R-modules which have rank > 0 at every $p \in \mathrm{Spec}(R)$. A well-known theorem of Bass states that

P is an object of $\mathrm{FP}(R)$ iff there exists a finitely generated projective R-module such that $P \otimes_R Q$ is free. Maps in $\mathrm{FP}(R)$ are module isomorphisms, the product is $\otimes_R$. The objects of $\mathrm{Az}(R)$ are Azumaya R-algebras; the product is $\otimes_R$. There is a functor End: $\mathrm{FP}(R) \to \mathrm{Az}(R)$ given by $P \mapsto \mathrm{End}_R(P)$, and this functor induces the exact sequences [4, pp. 117, 120] (cf. (2.3)).

(4.2a) $$1 \to U(R)/TU(R) \to K_1\mathrm{FP}(R) \to K_1\mathrm{Az}(R) \to t\,\mathrm{Pic}(R) \to 1$$

(4.2b) $$1 \to \mathrm{Pic}(R)/t\mathrm{Pic}(R) \to K_0\mathrm{FP}(R) \to K_0\mathrm{Az}(R) \to B(R) \to 1.$$

The proof of (4.1) will follow by a diagram chase from (4.2) and

(4.3) **Theorem.** *Let*

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \xrightarrow{f_2} & R_3 \end{array}$$

be a fibre product of rings in which f_2 is surjective. Then there are exact Mayer-Vietoris sequences:

$$K_1\mathrm{FP}(R) \to K_1\mathrm{FP}(R_1) \oplus K_1\mathrm{FP}(R_2) \to K_1\mathrm{FP}(R_3) \to K_0\mathrm{FP}(R)$$
$$\to K_0\mathrm{FP}(R_1) \oplus K_0\mathrm{FP}(R_2) \to K_0\mathrm{FP}(R_3)$$

and

$$K_1\mathrm{Az}(R) \to K_1\mathrm{Az}(R_1) \oplus K_1\mathrm{Az}(R_2) \to K_1\mathrm{Az}(R_3) \to K_0\mathrm{Az}(R)$$
$$\to K_0\mathrm{Az}(R_1) \oplus K_0\mathrm{Az}(R_2) \to K_0\mathrm{Az}(R_3).$$

Proof of (4.1). The functor End defines a map of cartesian squares of categories from

$$\begin{array}{ccc} \mathrm{FP}(R) & \to & \mathrm{FP}(R_1) \\ \downarrow & & \downarrow \\ \mathrm{FP}(R_2) & \to & \mathrm{FP}(R_3) \end{array}$$

to

$$\begin{array}{ccc} \mathrm{Az}(R) & \to & \mathrm{Az}(R_1) \\ \downarrow & & \downarrow \\ \mathrm{Az}(R_2) & \to & \mathrm{AZ}(R_3) \end{array}$$

So by VII (4.3) of [3] we have a map of Mayer-Vietoris sequences involving the two sequences of Theorem (4.3). Using the sequences of (4.2) we get a large diagram. We abbreviate groups in the diagram as follows: for a functor G, set

$G(R_i) = G(R_1) \oplus G(R_2)$; $FU(S) = U(S)/tU(S)$, $F\,\mathrm{Pic}(S) = \mathrm{Pic}(S)/t\,\mathrm{Pic}(S)$:

$$
\begin{array}{ccccccccccc}
1 \to & FU(R) & \longrightarrow & K_1\mathrm{FP}(R) & \longrightarrow & K_1\mathrm{Az}(R) & \longrightarrow & t\,\mathrm{Pic}(R) & \longrightarrow & 1 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 \to & FU(R_i) & \longrightarrow & K_1\mathrm{FP}(R_i) & \longrightarrow & K_1\mathrm{Az}(R_i) & \longrightarrow & t\,\mathrm{Pic}(R_i) & \longrightarrow & 1 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 \to & FU(R_3) & \longrightarrow & K_1\mathrm{FP}(R_3) & \longrightarrow & K_1\mathrm{Az}(R_3) & \longrightarrow & t\,\mathrm{Pic}(R_3) & \longrightarrow & 1 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 \to & F\,\mathrm{Pic}(R) & \longrightarrow & K_0\mathrm{FP}(R) & \longrightarrow & K_0\mathrm{Az}(R) & \longrightarrow & B(R) & \longrightarrow & 1 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 \to & F\,\mathrm{Pic}(R_i) & \longrightarrow & K_0\mathrm{FP}(R_i) & \longrightarrow & K_0\mathrm{Az}(R_i) & \longrightarrow & B(R_i) & \longrightarrow & 1 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 \to & F\,\mathrm{Pic}(R_3) & \longrightarrow & K_0\mathrm{FP}(R_3) & \longrightarrow & K_0\mathrm{Az}(R_3) & \longrightarrow & B(R_3) & \longrightarrow & 1
\end{array}
$$

The maps in the last column are induced by exactness of the rows. The first statement of (4.1) follows by chasing diagram (4.4). The last statement follows from the first statement by putting together the sequence of (1.10) and sequence (4.1a).

(4.5) **Remark.** If $B(R) = tH^2_{\mathrm{T}}(R, U)$ then (4.1) follows without (4.3) by considering the torsion parts and the cokernels of the torsion parts of sequence (1.10) and chasing the resulting diagram.

The remainder of §4 is devoted to a sketch of the proof of (4.3), which proof is essentially a verification of Remark (5.2), p. 481 of [3] for the categories **FP** and **Az**.

The proof of (4.3) is a matter of showing that a general theorem of Bass [3, VII, (4.3), p. 314] is applicable to our situation. We therefore recall some definitions and results from [3]. The notation $A \in \mathbf{C}$ means A is an object of **C**.

(4.6) **Definitions.** Let

(4.7)
$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{H_1} & \mathbf{C}_1 \\
{\scriptstyle H_2}\downarrow & & \downarrow{\scriptstyle F_1} \\
\mathbf{C}_2 & \xrightarrow{F_2} & \mathbf{C}_3
\end{array}
$$

be a diagram of categories with product and product preserving functors, and α: $F_1H_1 \to F_2H_2$ a natural isomorphism.

The functor F_2: $\mathbf{C}_2 \to \mathbf{C}_3$ is E-surjective if, given $A \in \mathbf{C}_2$ and β' in the commutator subgroup of $\mathrm{Aut}_{\mathbf{C}_3}(F_2A)$, there exists $B \in \mathbf{C}_2$ and β in the commutator subgroup of $\mathrm{Aut}_{\mathbf{C}_2}(A \perp B)$ so that $\beta' \perp 1_{F_2B} = F_2\beta$.

The functor F_2 is cofinal if given any $A' \in \mathbf{C}_3$ there exists $B' \in \mathbf{C}_3$, $A \in \mathbf{C}_2$ so that $A' \perp B' \cong F_2A$. The functor F_2 is cofinal with respect to F_1 if given $A_1 \in \mathbf{C}_1$ there exists $A_1' \in \mathbf{C}_1$, $A_2 \in \mathbf{C}_2$ such that $F_2A_2 \cong F_1(A_1 \perp A_1')$.

A basic object for $\mathbf{C}_2$ is an object A so that the inclusion functor from the category whose objects are A^n, $n \geq 1$, to $\mathbf{C}_2$, is cofinal. Given a basic object A in $\mathbf{C}_2$ such that $A^n \cong A^m$ implies $n = m$, set $G(A^\infty) = \varinjlim G(A^n)$ where $G(A^n) = \operatorname{Aut}_{\mathbf{C}_2}(A^n)$. Then $K_1\mathbf{C}_2 = G(A^\infty)/[G(A^\infty), G(A^\infty)]$ [4, p. 25].

(4.8) **Remark.** Suppose A is a basic object for $\mathbf{C}_2$, and F_2A is a basic object for $\mathbf{C}_3$. Then F_2 is cofinal, and F_2 is E-surjective iff the map

$$[G(A^\infty), G(A^\infty)] \longrightarrow [G((FA)^\infty), G((FA)^\infty)]$$

is onto.

(4.9) **Definitions, continued.** Given diagram (1), define the category $\mathbf{C}_1 \times_{\mathbf{C}_3} \mathbf{C}_2$ as follows:

The objects of $\mathbf{C}_1 \times_{\mathbf{C}_3} \mathbf{C}_2$ are triples (A_1, α, A_2), with $A_i \in \mathbf{C}_i$, $\alpha\colon F_1A_1 \xrightarrow{\cong} F_2A_2$; maps $(f_1, f_2)\colon (A_1, \alpha, A_2) \longrightarrow (B_1, \beta, B_2)$ are pairs of maps $f_i\colon A_i \longrightarrow B_i$ which commute with α and β. Define a functor $T\colon \mathbf{C} \longrightarrow \mathbf{C}_1 \times_{\mathbf{C}_3} \mathbf{C}_2$, by $T(A) = (H_1A, \alpha_A, H_2A)$.

(4.10) (Bass). The diagram (4.7) yields a Mayer-Vietoris sequence

$$K_1\mathbf{C} \longrightarrow K_1\mathbf{C}_1 \oplus K_1\mathbf{C}_2 \longrightarrow K_1\mathbf{C}_3 \longrightarrow K_0\mathbf{C} \longrightarrow K_0\mathbf{C}_1 \oplus K_0\mathbf{C}_2 \longrightarrow K_0\mathbf{C}_3$$

if

(a) F_1 and F_2 are cofinal and cofinal with respect to each other,

(b) F_2 is E-surjective, and

(c) the functor T is an equivalence.

Theorem (4.3) will follow from (4.10) applied to the diagram

(4.11)
$$\begin{array}{ccc} \mathbf{C}(R) & \xrightarrow{H_1} & \mathbf{C}(R_1) \\ {\scriptstyle H_2}\downarrow & & \downarrow{\scriptstyle F_1} \\ \mathbf{C}(R_2) & \xrightarrow{F_2} & \mathbf{C}(R_3) \end{array}$$

with $\mathbf{C} = \mathbf{FP}$ or $\mathbf{Az}$, where the functors are induced by "base change". The map $\alpha\colon F_1'H_1 \longrightarrow F_2H_2$ is induced by the isomorphism $(R \otimes_R R_1) \otimes_{R_1} R_3 \cong (R \otimes_R R_2) \otimes_{R_2} R_3$.

Proof of (4.3). (a) Cofinal subsets of $\mathbf{FP}(R)$, $\mathbf{Az}(R)$ are R^n, $M_n(R)$, respectively, for $n \geq 1$. Since F_i, $i = 1, 2$, takes cofinal sets to cofinal sets in each case, (a) is clear.

(b) *For* **FP**. We assumed that $f_2\colon R_2 \longrightarrow R_3$ is subjective. By (4.6) it is enough to show that $[GL_\otimes(R_2), GL_\otimes(R_2)] \longrightarrow [GL_\otimes(R_3), GL_\otimes(R_3)]$ is onto.

But if $\gamma \in [GL_n(R_3), GL_n(R_3)]$ $\gamma \otimes 1 \in E_{2n}(R_3)$ by the Whitehead lemma [3, V, (1.7), p. 226; cf. p. 519 and (1.9), p. 228]; $E_{2n}(R_2) \to E_{2n}(R_3)$ is clearly surjective, and $E_{2n}(R_2) \subseteq [GL_{2n}(R_2), GL_{2n}(R_2)]$ by [3, V, (1.5), p. 223].

(b) *For* Az. Recall $EA_n(R) = \mathrm{Aut}(\mathrm{End}_R(R^n))$, and $AE(R) = \varinjlim_n AE_n(R)$.

We must show that $[AE(R_2), AE(R_2)] \to [AE(R_3), AE(R_3)]$ is onto. *Notation*: $[G(R)] = [G(R), G(R)]$ in the next argument.

Clearly, $[GL_\otimes(R_i)] \to [PGL(R_i)]$ is onto for $i = 2, 3$. Since $[GL_\otimes(R_2)] \to [GL_\otimes(R_3)]$ is onto, it follows trivially that $[PGL(R_2)] \to [PGL(R_3)]$ is onto.

Now we have the commutative diagram, $i = 2, 3$.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & [PGL(R_i)] & \xrightarrow{j} & [AE(R_i)] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & PGL(R_i) & \longrightarrow & AE(R_i) & \longrightarrow & t\,\mathrm{Pic}(R_i) & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & PGL(R_i)/[PGL(R_i)] & \to & AE(R_i)/[AE(R_i)] & \to & t\,\mathrm{Pic}(R_i) & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

The bottom row is exact by [4, p. 119]. A diagram chase then shows that the map j is an isomorphism. So $[AE(R_2)] \to [AE(R_3)]$ is onto, and E-surjectivity is true for Az.

(c) The validity of (c) is the content of the remark [3, IX, (5.2), p. 481] of Bass. Verifying (c) along the lines of the proof of Milnor's theorem [3, IX, (5.1), p. 479] for FP is straightforward, so we omit it. If one follows the same proof for Az the only slightly nontrivial point is to show that if $(A_1, \alpha, A_2) = A$ is in $\mathrm{Az}(R_1) \times_{\mathrm{Az}(R_3)} \mathrm{Az}(R_2)$, that is $A_1 \in \mathrm{Az}(R_1)$, $A_2 \in \mathrm{Az}(R_2)$ and α: $F_1(A_1) \cong F_2(A_2)$ as R_3-algebras, then the fibre product $S(A_1, \alpha, A_2) = S(A)$ of the diagram

(4.12)
$$\begin{array}{ccccc}
S(A) & \longrightarrow & & & A_2 \\
\downarrow & & & & \downarrow \\
A_1 & \to & F_1(A_1) & \to & F_2(A_2)
\end{array}$$

is in $\mathrm{Az}(R)$.

One does this by tensoring the diagram (4.12) with the diagram for $A^0 = (A_1^0, A_2^0, \alpha^0)$ to get a commutative diagram which, after identifying $A_i \otimes A_i^0$ with $\mathrm{End}_{R_i}(A_i)$, becomes

(4.13)
$$\begin{array}{ccccc}
S(A) \otimes S(A^0) & \longrightarrow & & & \mathrm{End}_{R_2}(A_2) \\
\downarrow & & & & \downarrow \\
\mathrm{End}_{R_1}(A_1) & \to & F_1(\mathrm{End}_{R_1}(A_1)) & \xrightarrow{\alpha \otimes \alpha^0} & F_2(\mathrm{End}_{R_2}(A_2))
\end{array}$$

On the other hand, if B is the fibre product in **FP**:

$$\begin{array}{ccc} B & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 \to F_1A_1 & \to & F_2A_2 \end{array}$$

then $\mathrm{End}_R\, B$ is the fibre product of the corresponding diagram of endomorphism rings.

One shows that $S(A) \otimes S(A^0)$ is the fibre product in **P**, hence in **FP**, of the diagram (4.13). This follows by observing that $S(A) \otimes S(A^0)$ is described by a fibre product diagram like (4.13) except that the map $\alpha \otimes \alpha^0$ is replaced by some unknown function: this by Milnor's theorem [3, IX, (5.1)]; on the other hand $S(A) \otimes S(A^0)$ is isomorphic (by i) to an R-submodule of the fibre product $\mathrm{End}_R(B)$ of (4.13). Thus we have two exact sequences and a commutative diagram of R-modules:

$$\begin{array}{ccccccccc} 0 \to & S(A) \otimes S(A^0) & \to & \mathrm{End}_{R_1}(A_1) \otimes \mathrm{End}_{R_2}(A_2) & \xrightarrow{\gamma_1} & F_2(\mathrm{End}_{R_2}(A_2)) & \to 0 \\ & \downarrow i & & \| & & & \\ 0 \to & \mathrm{End}_R(B) & \longrightarrow & \mathrm{End}_{R_2}(A_1) \otimes \mathrm{End}_{R_2}(A_2) & \xrightarrow{\gamma_2} & F_2(\mathrm{End}_{R_2}(A_2)) & \to 0 \end{array}$$

where $\gamma_1 = (\alpha \otimes \alpha^0)F_1 - F_2$, and γ_2 is the corresponding map from the fibre product diagram for $\mathrm{End}_R(A)$. So there exists an epimorphism β from $F_2(\mathrm{End}_{R_2}(A_2))$ to itself so that $\gamma_1 = \beta\gamma_1$. But since $F_2(\mathrm{End}_{R_2}(A_2))$ is a finitely generated projective R_3-module β must be an isomorphism, hence i is an isomorphism onto $\mathrm{End}_R(B)$. By [2, Theorem 3.5] $S(A)$ is an Azumaya R-algebra.

The rest of the proof of condition (c) of (4.10) for Az, following the lines of the proof of Milnor's theorem in [3], is straightforward and will be omitted. That concludes the proof of (4.3).

Added in proof. Knus and Ojanguren have given in [18] a stronger version of (4.1).

REFERENCES

1. M. Artin, *Grothendieck topologies*, Harvard University Notes, 1962.

2. M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. 97 (1960), 367–409. MR 22 #12130.

3. H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968. MR 40 #2736.

4. ———, *Lectures on topics in algebraic K-theory*, Tata Institute of Fundamental Research Lectures on Math., no. 41, Tata Institute of Fundamental Research, Bombay, 1967. MR 43 #4885.

5. N. Bourbaki, *Eléments de mathématique*. Fasc. XXVII. *Algèbre commutative*. Chap. 2: *Localisation*, Actualités Sci. Indust., no. 1290, Hermann, Paris, 1961. MR 36 #146.

6. S. U. Chase and A. Rosenberg, *Amitsur cohomology and the Brauer group*, Mem. Amer. Math. Soc. No. 52 (1965), 34–79. MR 33 #4119.

7. E. Davis, *Rings of algebraic numbers and functions*, Math. Nachr. 29 (1965), 1–7. MR 31 #3411.

8. D. Dobbs, *On inflation and torsion in Amitsur cohomology*, Canad. J. Math. 24 (1972), 239–261.

9. G. Garfinkel, *Amitsur cohomology and an exact sequence involving Pic and the Brauer group*, Thesis, Cornell University, Ithaca, N. Y., 1968.

10. S. Gersten, *On Mayer-Vietoris functors and algebraic K-theory*, J. Algebra 18 (1971), 51–88. MR 43 #6290.

11. A. Grothendieck, *Le groupe de Brauer*. I, II, III, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam; Masson, Paris, 1968, pp. 46–188. MR 39 #5586 a, b, c.

12. R. Hoobler, *A generalization of the Brauer group and Amitsur cohomology*, Thesis, University of California, Berkeley, Calif., 1965.

13. M. Nagata, *Local rings*, Interscience Tracts in Pure and Appl. Math., no. 13, Interscience, New York, 1962. MR 27 #5790.

14. M. Raynaud, *Anneaux locaux henséliens*, Lecture Notes in Math., vol. 169, Springer-Verlag, Berlin and New York, 1970. MR 43 #3252.

15. A. Rosenberg and D. Zelinsky, *Automorphisms of separable algebras*, Pacific J. Math. 11 (1961), 1109–1117. MR 26 #6215.

16. J.-P. Serre, *Cohomologie galoisienne*, 3rd ed., Lecture Notes in Math., vol. 5, Springer-Verlag, Berlin and New York, 1965. MR 34 #1328.

17. M. Artin, *On the joins of Hensel rings*, Advances in Math. 7 (1971), 282–296. MR 44 #6690.

18. M.-A. Knus and M. Ojanguren, *A Mayer-Vietoris sequence for the Brauer group*, (preprint) (August, 1973).

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, ALBANY, NEW YORK 12222